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BRAUER FACTOR SETS AND SIMPLE ALGEBRAS

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ABSTRACT. It is shown that the Brauer factor set (c_{ijk}) of a finite-dimensional division algebra of odd degree n can be chosen such that $c_{iji} = c_{iij} = c_{jii} = 1$ for all i, j and $c_{ijk} = c_{kji}^{-1}$. This implies at once the existence of an element $a \neq 0$ with $\operatorname{tr}(a) = \operatorname{tr}(a^2) = 0$; the coefficients of x^{n-1} and x^{n-2} in the characteristic polynomial of a are thus 0. Also one gets a generic division algebra of degree n whose center has transcendence degree $n + (n-1)(n-2)/2$, as well as a new (simpler) algebra of generic matrices. Equations are given to determine the cyclicity of these algebras, but they may not be tractable.

Introduction. The theory of Brauer factor sets, dormant for 40 years after the work of Brauer [**2**, **3**], has been revived in the excellent account of Jacobson [**5**]. In this paper we add some observations to introduce a slightly simpler form to Brauer's factor sets, thereby adding insights to the question of cyclicity of division algebras of odd degree. Of course, in view of the theorem of Amitsur [**1**] the only open case is when the degree n is a product of distinct primes. In this case, by a theorem of Albert (cf. [**6**, Theorem 3.2.3]) the cyclicity is equivalent to the existence of an element whose nth power is central and all of whose smaller powers are noncentral, and this formulation will be a principal object of our study. Although the methods presented here permit a concrete combinatorial formulation, the resulting equality still seems to be very difficult to handle.

In the course of the investigations we come across a new algebra which has a simpler structure than the algebra of generic matrices but which is closely tied in with the theory of division algebras, and may have other applications. In what follows, F is a field and R is a central simple F-algebra of degree n, i.e. $[R:F] = n^2$. Some basic familiarity with the theory of simple algebras might be useful.

1. Brauer factor sets. Let us review briefly the main results of [**5**] concerning Brauer factor sets. Suppose K is a maximal separable subfield of R over F (so $[K:F] = n$) and E is the normal closure of K, with $G = \operatorname{Gal}(E/F)$. Then $K = F[u]$ for some u in K, implying the minimal polynomial $f(\lambda)$ of u has degree n, and E is the splitting field of f over K. Thus we can write $f = \prod_{i=1}^{n}(\lambda - r_i)$ for suitable r_i in E. Then G permutes the r_i by acting on the subscripts ($\sigma r_i = r_{\sigma i}$), and is thereby viewed as a transitive subgroup of the symmetric group $\operatorname{Sym}(n)$.

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There exists v in R such that $R = KvK$, and one can view naturally $R \subset R \otimes_F K \approx M_n(K) \subseteq M_n(E)$. Picking a set of matric units e_{ij} such that $u = \sum_{i=1}^n r_i e_{ii}$, we have $v = (v_{ij}) \in M_n(E)$ where each $v_{ij} \neq 0$. Let $c_{ijk} = v_{ij}v_{jk}v_{ik}^{-1}$. Then $\{c_{ijk}: 1 \leqslant i, j, k \leqslant n\}$ satisfies the following conditions, called the *Brauer factor set conditions* for all i, j, k, m and all σ in G.

(i) $\sigma c_{ijk} = c_{\sigma i, \sigma j, \sigma k}$.

(ii) $c_{ijk}c_{ikm} = c_{ijm}c_{jkm}$.

Note from (ii) that for $i = j$ and $m = 1$, we get $c_{iik} = c_{ii1}$ for all i, k. Likewise, $k = j$ yields $c_{ijj} = c_{jjm}$.

Conversely, a set of n^3 elements (c_{ijk}) from E satisfying the Brauer factor set conditions is called a *Brauer factor set*, and defines an associative multiplication on the F-vector subspace $R' = \{(a_{ij}) \in M_n(E): \sigma a_{ij} = a_{\sigma i, \sigma j}$ for all σ in $G\}$ by the rule

$$(a_{ij})(b_{ij}) = \sum_{j=1}^{n} (a_{ij}c_{ijk}b_{jk})e_{ik}.$$

R' is then a simple F-algebra which can be injected into $M_n(E)$ via the map $(a_{ij}) \to \Sigma(c_{ij1}a_{ij})e_{ij}$. Moreover, suppose (c_{ijk}) and (c'_{ijk}) are Brauer factor sets with respect to the same K (and E). The ensuing simple algebras are isomorphic iff c_{ijk} and c'_{ijk} are *equivalent*, by which one means there are w_{ij} in E such that

$$\sigma w_{ij} = w_{\sigma i, \sigma j} \quad \text{and} \quad c'_{ijk} = w_{ij}w_{jk}w_{ik}^{-1}c_{ijk}$$

for all σ in G and all i, j, k. In this case we write $(c_{ijk}) \sim (c'_{ijk})$. Write (1) for the trivial Brauer set, all of whose entries are 1; this gives rise to the matric algebra $M_n(F)$; cf. [**5**, Theorem 3.1]. We also need the following tensor product theorem; cf. [**5**, Theorem 3.6]:

Suppose R, R' correspond, respectively, to the Brauer factor sets (c_{ijk}), (c'_{ijk}) over K, and let $c''_{ijk} = c_{ijk}c'_{ijk}$. Then (c''_{ijk}) *is a Brauer factor set corresponding to some simple algebra* R'', *and* $R \otimes_F R' \approx M_n(R'')$.

Implicit in [**5**] is the way to recapture v from the matrix presentation in terms of a given Brauer factor set (c_{ij}). Indeed, taking v as the element corresponding to the matrix all of whose entries are 1, we have $v \in R$ by [**5**, Theorem 2.14]. The element u corresponds to the diagonal matrix whose entries are $c_{ii1}^{-1}r_i$, and then $u^k v u^m$ is the matrix $(c_{ij1}^{-1}a_{ij})$, where $a_{ij} = r_i^k r_j^m$. These elements span R over F by [**5**, Lemma 2.8], so we conclude $R = KvK$, as desired.

Note that v, as constructed above, was with respect to the multiplication induced by the Brauer factor set; if we want to view R as a subalgebra of $M_n(E)$ then we must multiply every entry by c_{ij1}, i.e. v is identified with the matrix (c_{ij1}).

Having concluded the preliminary survey, let us add one small observation. R^{op} denotes the opposite algebra, i.e. $R \otimes_F R^{\text{op}} \approx M_{n^2}(F)$.

PROPOSITION 1. *If* (c_{ijk}) *is a Brauer factor set corresponding to* R *then* (c_{kji}) *corresponds to* R^{op}.

PROOF. Let $w_{ij} = c_{iji}c_{ii1}$. If $k = \sigma i$ and $m = \sigma j$ then

$$\sigma w_{ij} = c_{kmk}c_{k,k,\sigma 1} = c_{kmk}c_{kk1} = w_{km}.$$

Moreover, by the second Brauer factor set condition, taking $m = j$ yields

$$w_{ij} = c_{iji}c_{iim} = c_{ijm}c_{jim}.$$

likewise, taking $k = i$ yields

$$c_{ijm} = c_{jim} = c_{iji}c_{iim} = w_{ij}.$$

Hence

$$w_{ij}w_{jk}w_{ik}^{-1} = (c_{ijk}c_{jik})(c_{jki}c_{kji})(c_{jik}c_{jki})^{-1} = c_{ijk}c_{kji},$$

proving $(c''_{ijk} = c_{ijk}c_{kji})$ is equivalent to (1), i.e. corresponds to $M_n(F)$, so (c_{kji}) corresponds to R^{op} by the tensor product theorem. Q.E.D.

Note. The second Brauer factor set condition can be rewritten suggestively as

$$c_{ijk} = c_{ijm}c_{jkm}c_{ikm}^{-1}.$$

REMARK 2. If (c_{ijk}) is a Brauer factor set, then $(c_{ijk}^{t}) \sim (c_{ijk}^{t+mn})$ for all t, m in $\mathbf{Z}^{+}$.

PROOF. This follows at once from the tensor product theorem and the structure of simple algebras, but there is a direct computational proof of interest (which thereby gives an independent proof, using Brauer factor sets, that the Brauer group is torsion). Indeed, it suffices to show $(1) \sim (c_{ijk}^{n})$. But letting $w_{ij} = \prod_{m=1}^{n} c_{ijm}$ yields

$$w_{ij}w_{jk}w_{ik}^{-1} = \prod_{m=1}^{n} c_{ijm}c_{jkm}c_{ikm}^{-1} = \prod_{m=1}^{n} c_{ijk} = c_{ijk}^{n},$$

as desired. Q.E.D.

DEFINITION 3. The Brauer factor set (c_{ijk}) is *normalized* if $c_{iij} = c_{iji} = c_{jii} = 1$ and $c_{kji} = c_{ijk}^{-1}$ for all i, j, k.

THEOREM 4. *For any Brauer factor set (c_{ijk}) there is a normalized Brauer factor set equivalent to (c_{ijk}^{2}). In particular, if n is odd then every Brauer factor set has an equivalent normalized Brauer factor set.*

PROOF. Let $c'_{ijk} = c_{ijk}c_{kji}^{-1}$. Then $(c'_{ijk}) \sim (c_{ijk}^{2})$ by Proposition 1. Moreover, (c'_{ijk}) is easily checked to be normalized. If n is odd, then letting $c''_{ijk} = (c'_{ijk})^{(n+1)/2}$, we see (c''_{ijk}) is normalized and equivalent to $((c_{ijk}^{2})^{(n+1)/2}) = (c_{ijk}^{n+1}) \sim (c_{ijk})$. Q.E.D.

COROLLARY 5. *If n is odd, then any simple algebra R of degree n has an element $a \neq 0$ with* $\mathrm{tr}(a) = \mathrm{tr}(a^2) = 0$, *where* tr *denotes the reduced trace.*

PROOF. Pick a normalized Brauer factor set (c_{ijk}). Then for any $a = (a_{ij})$ in R, written with respect to (c_{ijk}), we have

$$\mathrm{tr}(a) = \sum_{i=1}^{n} a_{ii}c_{ii1} = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad \mathrm{tr}(a^2) = \sum_{i,j=1}^{n} a_{ij}c_{iji}a_{ji}c_{ii1} = \sum_{i,j=1}^{n} a_{ij}a_{ji}.$$

But in $M_n(F)$ there is certainly a matrix $b \neq 0$ with $\mathrm{tr}(b) = \mathrm{tr}(b^2) = 0$. Representing b with respect to the trivial Brauer factor set (1) gives a matrix (a_{ij}) with $a_{\sigma i, \sigma j} = a_{ij}$ and $\Sigma a_{ii} = 0 = \Sigma a_{ij}a_{ji}$; this is the respresentation of our desired element a with respect to (c_{ijk}) because (c_{ijk}) is normalized. Q.E.D.

REMARK 6. Suppose $\mathrm{char}(F) = 0$ for convenience. If $\mathrm{tr}(a) = \mathrm{tr}(a^2) = \cdots = \mathrm{tr}(a^{n-1}) = 0$, then by Newton's formulas all the characteristic coefficients of a

vanish except possibly for the determinant. Thus the minimal polynomial of a is λ^n-det(a), so $a^n \in F$. Conversely for n prime, if $a^n \in F$ and $a^i \notin F$ for all $i < n$, then the characteristic coefficients are 0, implying $\operatorname{tr}(a) = \operatorname{tr}(a^2) = \cdots = \operatorname{tr}(a^{n-1}) = 0$.

In our case $\operatorname{tr}(a) = \operatorname{tr}(a^2) = 0$ implies the coefficients of x^{n-1} and x^{n-2} of the (reduced) characteristic polynomial are 0.

In particular, for $n = 3$ we have reproved the theorem of Wedderburn [**8**] that there is $a \notin F$ with $a^3 \in F$. It may be of interest to write $a = (a_{ij})$ more explicitly. Indeed we take $a_{ii} = 0$ for all i and $a_{ij} = a_i$ for all $j \neq i$ where

$$a_i = (2r_i - r_{i+1} - r_{i+2})^{-1},$$

subscripts modulo 3. Thus $\operatorname{tr}(a) = 0 + 0 + 0 = 0$ and

$$\operatorname{tr} a^2 = a_1a_2 + a_1a_3 + a_2a_3 = \operatorname{tr}\left(\left((2r_1 - r_2 - r_3)^{-1}\right)^{-1}\right) = \operatorname{tr}(2r_1 - r_2 - r_3) = 0.$$

(The first two uses of "tr" are the reduced trace; the last two are the trace of the field extension K/F.)

This computation raises several significant questions for $n > 3$:

(i) Can we take a so as to have its diagonal entries to be 0?

(ii) Can we take a so that also $\operatorname{tr}(a^3) = 0$?

Digression. If we change the maximal subfield K then the first question has a positive answer. Indeed, choose a such that $\operatorname{tr}(a) = \operatorname{tr}(a^2) = 0$ and a is separable of degree n over F. (This is easily seen to be possible in view of the general construction of the element in the proof of Corollary 5.) Then replacing K by $F[a]$ we can use the matrix whose diagonal entries are 0 and whose entry in the 1-2 position is a. (This determines each entry in view of the conjugacy conditions.)

Of course the second question is very important, in view of Remark 6, and can be written as follows, since $\operatorname{tr}(a^3) = \Sigma(a_{ij}c_{ijk}a_{jk})c_{iki}a_{ki}c_{ii1}$:

Question A. Can one choose $a = (a_{ij})$ such that $\sigma a_{ij} = a_{\alpha i, \sigma j}$ for all σ in G and $\Sigma a_{ij}a_{jk}a_{ki}c_{ijk}c_{iki}c_{ii1} = 0$?

REMARK 7. Choosing (c_{ijk}) normalized (cf. Theorem 4), i.e. $c_{iij} = c_{iji} = c_{jii} = 1$ and $c_{kji} = c_{ijk}^{-1}$, we also have $c_{ijk} = c_{jki} = c_{kij}$ for all i, j, k. (Indeed by the note preceding Remark 2, taking $m = j$ we have

$$c_{ijk} = c_{ijj}c_{jkj}c_{ikj}^{-1} = c_{ikj}^{-1} = c_{jki},$$

as desired.)

REMARK 8. In view of Remark 7, normalizing the (c_{ijk}) enables us to modify the equation of Question A as

$$0 = \sum_{i=1}^{n} a_{ii}^3 + 3\sum_{i<j} a_{ij}a_{ji}(a_{ii} + a_{jj}) + 3\sum_{i<j<k}\left(a_{ij}a_{jk}a_{ki}c_{ijk} + a_{ik}a_{kj}a_{ji}c_{ijk}^{-1}\right).$$

Note that the c_{ijk} in the last summand cannot be removed, and I suspect they prevent a general solution. We shall discuss this further in §3.

2. Generic constructions. Brauer's method gains significance when confronted with generic matrices, an invention of Amitsur in polynomial identity theory. Let ξ denote a set of $n^2 + n$ commuting indeterminates, denoted as ξ_{ij} and ξ'_{ii} for $\leqslant i, j \leqslant n$, over a field F_0 and let $F_0\{Y_1, Y_2\}$ be the F_0-subalgebra of $M_n(F_0[\xi])$ generated by $Y_1 = \sum_{i=1}^n \xi'_{ii}e_{ii}$ and $Y_2 = \sum_{i,j} \xi_{ij}e_{ij}$. Then $F_0\{Y_1, Y_2\}$ is a prime ring of PI-class n, cf. [**6**, Proposition 2.4.11], and localizing by inverting all nonzero central elements gives us a simple F_0-algebra by [**6**, Theorem 1.7.9], which is called $\mathrm{UD}(F_0, n)$; in fact, $\mathrm{UD}(F_0, n)$ is a division algebra by a theorem of Amitsur [**6**, Theorem 3.2.6]. Procesi proved the center of $\mathrm{UD}(F_0, n)$ has transcendence degree $n^2 + 1$ over F_0, and we need his calculations here. Replace Y_2 by the matrix Y'_2, obtained by conjugating Y_2 by the diagonal matrix $e_{11} + \sum_{i=2}^n \xi_{1i}e_{ii}$. Then

$$Y'_2 = \xi_{11}e_{11} + \sum_{j=2}^{n} e_{1j} + \sum_{i=2}^{n} \xi_{1i}\xi_{i1}e_{i1} + \sum_{i,j=2}^{n} \xi_{1i}\xi_{ij}\xi_{1j}^{-1}e_{ij}.$$

Letting c_{ij} denote the i-j coefficient of Y'_2 we see that

$$\begin{aligned} c_{ii} &= \xi_{ii}, \\ c_{1j} &= 1 \quad \text{for all } j \neq 1, \\ c_{i1} &= \xi_{1i}\xi_{i1} \quad \text{for all } i \neq 1, \\ c_{ij} &= \xi_{1i}\xi_{ij}\xi_{1j}^{-1} \quad \text{for all } i,\ j \geqslant 2. \end{aligned}$$

By [**6**, Theorem 1.10.28] those $c_{ij} \neq 1$ are algebraically independent over $K_0 = F_0(\xi'_{11} \cdots \xi'_{nn})$. We replace $F_0\{Y_1, Y_2\}$ by its isomorphic copy $F_0\{Y_1, Y'_2\}$ whose ring of central quotients is $R \approx \mathrm{UD}(F_0, n)$ with center F. (Incidentally, one usually takes $F_0 = \mathbf{Q}$ or $F_0 = \mathbf{Q}(\sqrt[n]{1})$.)

$\mathrm{Sym}(n)$ acts on the ξ_{ij} and ξ'_{ii} by the natural action of these subscripts, and, in view of [**6**, Theorem 3.3.31], this action restricts to a group of automorphisms on the field $E = K_0(c_{ij}\colon 1 \leqslant i,\ j \leqslant n)$ whose fixed subfield is F. (This is the reason F has transcendence degree $n^2 + 1$ over F.) Let K be the subfield of E fixed under $G_1 = \{\sigma \in \mathrm{Sym}(n)\colon \sigma 1 = 1\} \approx \mathrm{Sym}(n-1)$. Then E is the normal closure of K. Moreover, $[K:F] = n = [F(Y_1):F]$; there is a homomorphism $F(Y_1) \to K$, given by $\sum \alpha_i Y_1^i \to \sum \alpha_i(\xi'_{11})^i$, which is thus an isomorphism, implying $K \approx F(Y_1)$.

Taking $v = Y'_2$ we get the Brauer factor set (c_{ijk}), where $c_{ij}c_{jk}c_{ik}^{-1}$ and, in fact, a routine computation shows

$$c_{ijk} = \xi_{ij}\xi_{jk}\xi_{ik}^{-1} \quad \text{for all } i,\ j,\ k.$$

Note that the usual specialization arguments make this example generic for Question A, in the sense that a positive answer in the special case would imply a positive answer in general. We can make the following further simplification.

Write $R \otimes_F R = M_n(R_1)$. By Theorem 4 we can display R_1 in terms of the normalized Brauer factor set (c''_{ijk}) where $c''_{ijk} = c_{ijk}c_{kji}^{-1}$. Moreover, for n odd, the structure of R is determined by the structure of R_1. (In particular, R is cyclic iff R_1 is cyclic.) Thus we are interested in exploiting the simpler form of (c''_{ijk}).

Note that $c_{ijk}c_{kji} = c_{iji}c_{ii1}$; hence $c''_{ijk} = c_{ijk}^2c_{iji}^{-1}c_{ii1}^{-1}$, and the algebraic independence of the c_{1ij} imply the c''_{1ij} are also algebraically independent over F_0. Letting $E'' = K_0(c''_{1ij}\colon 1 < i < j)$ and letting F'' (resp. K'') be the fixed subfield under the induced action of $\mathrm{Sym}(n)$ (resp. $\mathrm{Sym}(n-1)$), we see that (c''_{ijk}) induces a simple algebra R'' of degree n with center F'', which is, in fact, a (division) subalgebra of R_1, so $R_1 \approx R'' \otimes_{F''} F$.

Cyclicity of R'' would certainly imply cyclicity of R_1, but, on the other hand R'' has a simpler structure; for example F'' has transcendence degree

$$n + (n-1)(n-2)/2$$

over F_0.

As in §1 we have R'' is the E''-subalgebra of $M_n(E'')$ generated by K_0 and the matrix $v'' = (c''_{ij1})$. Thus, letting R' be the F_0-subalgebra generated by Y_1 and v'', we see R'' is a central extension of R', proving R' is a prime Pl-algebra of Pl-class n which, for n odd, is generic with respect to the cyclicity question.

R' is generated over F_0 by the matrices $Y_1 = \sum_{i=1}^n \xi'_{ii} e_{ii}$ and $v'' = \sum_{i,j=1}^n c''_{ij1} e_{ij}$. But $c''_{ii1} = c''_{1i1} = c''_{i11} = 1$, proving the first row, first column and diagonal entries of v'' are all 1. Moreover, for $i > n$ we have $c''_{ji1} = c''^{-1}_{ij1}$ by Remark 7, so clearly the $\{c''_{ij1}: i > j\}$ are algebraically independent over F_0. This discussion can be summarized in the next definition and theorem.

Definition 9. Let Y'_2 be the matrix whose entries in the first row, column and diagonal are all 1, and whose i-j entry is ξ_{ij} (resp. ξ_{ij}^{-1}) when $1 < i < j$ (resp. $1 < j < i$). The algebra of *modified generic matrices* is the F_0-subalgebra of $M_n(F_0[\xi])$ generated by $Y_1 = \sum_{i=1}^n \xi_{ii} e_{ii}$ and Y'_2.

Theorem 10. *The algebra of modified generic matrices is a domain of* Pl-*class n. If its (division) algebra of central quotients is cyclic then $A \otimes_{Z(A)} A$ is cyclic for every simple F_0-algebra A of degree n. (For n odd these statements are each equivalent to every simple F_0-algebra of degree n being cyclic.)*

Proof. The algebra of central quotients is isomorphic to the division algebra R' described preceding Definition 9, since the c''_{ij1} are algebraically independent; R' has the desired properties, as proved above. Q.E.D.

Note. If one starts with the algebra of modified generic matrices and builds the Brauer factor set from Y'_2 (putting $\xi_{ji} = \xi_{ij}^{-1}$ for $1 < i < j$), we get $c_{ijk} = \xi_{ij}\xi_{jk}\xi_{ki}$ whenever i, j, k are distinct, and $c_{ijk} = 1$ otherwise. Thus the (c_{ijk}) are normalized.

If we were to "know" the structure of the algebra of modified generic matrices then we would know the structure of all algebras of the form $R \otimes R$, which essentially would reduce the structure theory of finite-dimensional division algebras to the exponent 2 case. This has been studied by Jacobson [**5**, Theorem 6.4], who proved that in this case one can always take the (c_{ijk}) to be symmetric, i.e. $c_{ijk} = c_{kji}$, and, conversely, such a Brauer factor set implies the algebra has exponent 2. This setting actually has a generic solution arising from the theory of polynomial identities of rings with involution, yielding Jacobson's theorem as a consequence.

Namely, letting $^{(t)}$ denote the transpose of matrices, let $S = F_0\{Y_1, Y_2, Y_2^t\}$, where we recall $Y_1 = \sum \xi'_{ii} e_{ii}$ and $Y_2 = \sum_{i,j} \xi_{ij} e_{ij}$. Taking $v = Y_2 + Y_2^t$, we have

$$v = \sum_{i=1}^{n} \xi''_{ii} e_{ii} + \sum_{i<j} \xi''_{ij}(e_{ij} + e_{ji}),$$

where we define $\xi''_{ij} = \xi_{ij} + \xi_{ji}$ for all i, j. Note $\xi''_{ij} = \xi''_{ji}$. Hence,

$$c_{ijk} = \xi''_{ij}\xi''_{jk}(\xi''_{ik})^{-1} = \xi''_{jk}\xi''_{ij}(\xi''_{ik})^{-1} = \xi''_{kj}\xi''_{ji}(\xi''_{ki})^{-1} = c_{kji},$$

and the corresponding Brauer factor set is symmetric.

Actually, I think it may turn out to be more rewarding to use the *symplectic* involution, but the results are more technical and would be too much of a digression here.

3. Suggested application to the structure of division algebras. The prospect of applications to the structure theory is very tantalizing, because whereas it seems to give hope to answering the cyclicity question, the ensuing equations have proved intractable for me so far.

REMARK 11. As noted in §1, an algebra of (odd) prime degree n is cyclic iff there is an element a whose minimal polynomial is λ^n-$\det(a)$, i.e. $\mathrm{tr}(a) = \mathrm{tr}(a^2) = \cdots = \mathrm{tr}(a^{n-1}) = 0$, and it is possible to find a with $\mathrm{tr}(a) = \mathrm{tr}(a^2) = 0$. One is led to see whether, possibly, $\mathrm{tr}(a^3) = 0$, and the obvious test case is the division ring of Theorem 10, described as follows in terms of Brauer factor sets:

$$E_0 = \mathbf{Q}(\xi_{ij}\colon 1 < i < j \leqslant n \text{ or } 1 \leqslant i = j \leqslant n).$$

Put $\xi_{ji} = \xi_{ij}^{-1}$ for $1 < i < j \leqslant n$ and define $c_{ijk} = 1$ unless i, j, k are distinct, in which case $c_{ijk} = \xi_{ij}\xi_{jk}\xi_{ki}$. Letting $\mathrm{Sym}(n)$ act naturally on the subscripts, let E be the subfield of E_0 generated by the c_{1ij} for $1 < i < j$ and the ξ_{ii}; in fact, this contains all c_{ijk} since $c_{1ji} = c_{1ij}^{-1}$ and $c_{ijk} = c_{1ij}c_{1jk}c_{1ki}$. Let K be the subfield of E fixed by all σ in $\mathrm{Sym}(n)$ s.t. $\sigma 1 = 1$. We build our example R from the Brauer factor set (c_{ijk}) with respect to K and E. Then the existence of $a \neq 0$ in A with $\mathrm{tr}(a^3) = 0$ is equivalent to the existence of elements a_{ij} in E, not all 0, satisyfing $\sigma a_{ij} = a_{\sigma i, \sigma j}$ for all σ in $\mathrm{Sym}(n)$ as well as the equation given in Remark 8. In attempting to solve this equation (or, better yet, prove there is no solution) one may get rid of the ξ_{ii} by suitable specialization, so we may assume $E = \mathbf{Q}(c_{1ij}\colon 1 < i < j)$.

A sizable step would be accomplished if we could reduce the solution to the case $a_{ii} = 0$ for all i, which incidentally has been the case in all positive results so far obtained using Brauer factor sets (cf. second paragraph of Remark 6, as well as [**3**]). (I do not expect this can be achieved through conjugation by an element, since for $n = 2$ there exist elements of $\mathrm{UD}(F_0, 2)$ which, when written with respect to the *nonnormalized* (c_{ijk}), cannot be diagonalized, for example

$$\begin{pmatrix} \xi_{11} - \xi_{22} & 1 \\ 1 & \xi_{22} - \xi_{11} \end{pmatrix};$$

the idea behind this example came in a letter by Formanek. Nevertheless, I have not been able to provide a counterexample for $n = 3$, and thus have no counterexample for normalized Brauer sets.) Nevertheless, these methods seem to give real hope for a computational solution to the cyclicity question for $n = 5$.

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